

## 1 ls.mixed: Mixed effects Linear Regression

Use multi-level linear regression if you have covariates that are grouped according to one or more classification factors and a continuous dependent variable.

While generally called multi-level models in the social sciences, this class of models is often referred to as mixed-effects models in the statistics literature and as hierarchical models in a Bayesian setting. This general class of models consists of linear models that are expressed as a function of both *fixed effects*, parameters corresponding to an entire population or certain repeatable levels of experimental factors, and *random effects*, parameters corresponding to individual experimental units drawn at random from a population.

### 1.0.1 Syntax

```
z.out <- zelig(formula= y ~ x1 + x2 + tag(z1 + z2 | g),
               data=mydata, model="lm.multi")

z.out <- zelig(formula= list(mu=y ~ x1 + x2 + tag(z1, gamma | g),
                             gamma= ~ tag(w1 + w2 | g)), data=mydata, model="lm.multi")
```

### 1.0.2 Inputs

`zelig()` takes the following arguments for `multi`:

- **formula**: a two-sided linear formula object describing the systematic component of the model, with the response on the left of a `~` operator and the fixed effects terms, separated by `+` operators, on the right. Any random effects terms are included with the notation `tag(z1 + ... + zn | g)` with `z1 + ... + zn` specifying the model for the random effects and `g` the grouping structure. Random intercept terms are included with the notation `tag(1 | g)`.

Alternatively, **formula** may be a list where the first entry, **mu**, is a two-sided linear formula object describing the systematic component of the model, with the response on the left of a `~` operator and the fixed effects terms, separated by `+` operators, on the right. Any random effects terms are included with the notation `tag(z1, gamma | g)` with `z1` specifying the individual level model for the random effects, `g` the grouping structure and `gamma` references the second equation in the list. The **gamma** equation is one-sided linear formula object with the group level model for the random effects on the right side of a `~` operator. The model is specified with the notation `tag(w1 + ... + wn | g)` with `w1 + ... + wn` specifying the group level model and `g` the grouping structure.

### 1.0.3 Additional Inputs

In addition, `zelig()` accepts the following additional arguments for model specification:

- **data**: An optional data frame containing the variables named in **formula**. By default, the variables are taken from the environment from which **zelig()** is called.
- **family**: A GLM family, see **glm** and **family** in the **stats** package. If **family** is missing then a linear mixed model is fit; otherwise a generalized linear mixed model is fit. In the later case only **gaussian** family with "log" link is supported at the moment.
- **na.action**: A function that indicates what should happen when the data contain **NA**s. The default action (**na.fail**) causes **zelig()** to print an error message and terminate if there are any incomplete observations.

Additionally, users may wish to refer to **lmer** in the package **lme4** for more information, including control parameters for the estimation algorithm and their defaults.

#### 1.0.4 Examples

##### 1. Basic Example with First Differences

Attach sample data:

```
> data(voteincome)
```

Estimate model:

```
> z.out1 <- zelig(income ~ education + age + female + tag(1 | state), data=voteincome,
```

Summarize regression coefficients and estimated variance of random effects:

```
> summary(z.out1)
```

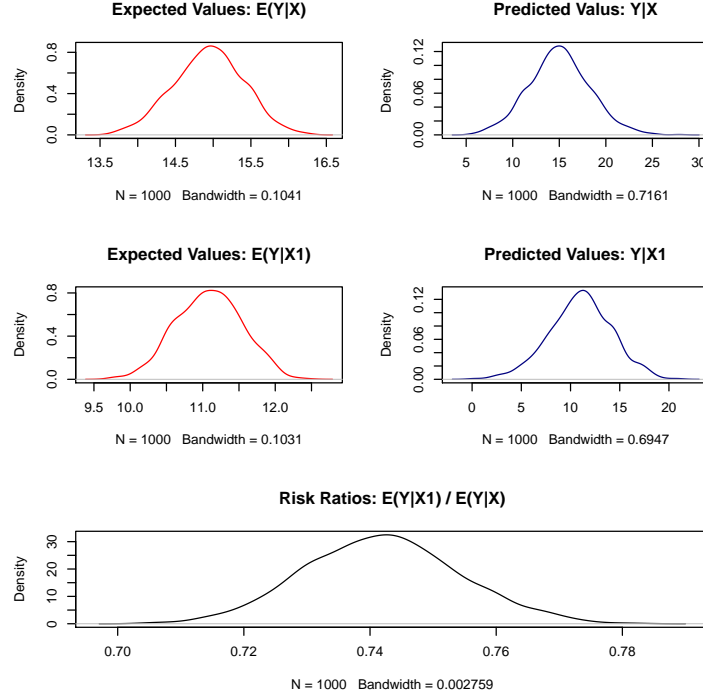
Set explanatory variables to their default values, with high (80th percentile) and low (20th percentile) values for education:

```
> x.high <- setx(z.out1, education=quantile(voteincome$education, 0.8))
> x.low <- setx(z.out1, education=quantile(voteincome$education, 0.2))
```

Generate first differences for the effect of high versus low education on income:

```
> s.out1 <- sim(z.out1, x=x.high, x1=x.low)
> summary(s.out1)
```

```
> plot(s.out1)
```



### 1.0.5 Mixed effects linear regression model

Let  $Y_{ij}$  be the continuous dependent variable, realized for observation  $j$  in group  $i$  as  $y_{ij}$ , for  $i = 1, \dots, M$ ,  $j = 1, \dots, n_i$ .

- The *stochastic component* is described by a univariate normal model with a vector of means  $\mu_{ij}$  and scalar variance  $\sigma^2$ .

$$Y_{ij} \sim \text{Normal}(y_{ij} | \mu_{ij}, \sigma^2)$$

- The  $q$ -dimensional vector of *random effects*,  $b_i$ , is restricted to be mean zero, and therefore is completely characterized by the variance covariance matrix  $\Psi$ , a  $(q \times q)$  symmetric positive semi-definite matrix.

$$b_i \sim \text{Normal}(0, \Psi)$$

- The *systematic component* is

$$\mu_{ij} \equiv X_{ij}\beta + Z_{ij}b_i$$

where  $X_{ij}$  is the  $(n_i \times p \times M)$  array of known fixed effects explanatory variables,  $\beta$  is the  $p$ -dimensional vector of fixed effects coefficients,  $Z_{ij}$  is the  $(n_i \times q \times M)$  array of known random effects explanatory variables and  $b_i$  is the  $q$ -dimensional vector of random effects.

### 1.0.6 Quantities of Interest

- The predicted values (**qi\$pr**) are draws from the normal distribution defined by mean  $\mu_{ij}$  and variance  $\sigma^2$ ,

$$\mu_{ij} = X_{ij}\beta + Z_{ij}b_i$$

given  $X_{ij}$  and  $Z_{ij}$  and simulations of  $\beta$  and  $b_i$  from their posterior distributions. The estimated variance covariance matrices are taken as correct and are themselves not simulated.

- The expected values (**qi\$ev**) are averaged over the stochastic components and are given by

$$E(Y_{ij}|X_{ij}) = X_{ij}\beta.$$

- The first difference (**qi\$fd**) is given by the difference in expected values, conditional on  $X_{ij}$  and  $X'_{ij}$ , representing different values of the explanatory variables.

$$FD(Y_{ij}|X_{ij}, X'_{ij}) = E(Y_{ij}|X_{ij}) - E(Y_{ij}|X'_{ij})$$

- In conditional prediction models, the average predicted treatment effect (**qi\$att.pr**) for the treatment group is given by

$$\frac{1}{\sum_{i=1}^M \sum_{j=1}^{n_i} t_{ij}} \sum_{i=1}^M \sum_{j:t_{ij}=1}^{n_i} \{Y_{ij}(t_{ij}=1) - \widehat{Y_{ij}(t_{ij}=0)}\},$$

where  $t_{ij}$  is a binary explanatory variable defining the treatment ( $t_{ij} = 1$ ) and control ( $t_{ij} = 0$ ) groups. Variation in the simulations is due to uncertainty in simulating  $Y_{ij}(t_{ij} = 0)$ , the counterfactual predicted value of  $Y_{ij}$  for observations in the treatment group, under the assumption that everything stays the same except that the treatment indicator is switched to  $t_{ij} = 0$ .

- In conditional prediction models, the average expected treatment effect (**qi\$att.ev**) for the treatment group is given by

$$\frac{1}{\sum_{i=1}^M \sum_{j=1}^{n_i} t_{ij}} \sum_{i=1}^M \sum_{j:t_{ij}=1}^{n_i} \{Y_{ij}(t_{ij}=1) - E[Y_{ij}(t_{ij}=0)]\},$$

where  $t_{ij}$  is a binary explanatory variable defining the treatment ( $t_{ij} = 1$ ) and control ( $t_{ij} = 0$ ) groups. Variation in the simulations is due to uncertainty in simulating  $E[Y_{ij}(t_{ij} = 0)]$ , the counterfactual expected value of  $Y_{ij}$  for observations in the treatment group, under the assumption that everything stays the same except that the treatment indicator is switched to  $t_{ij} = 0$ .

- If "log" link is used, expected values are computed as above and then exponentiated, while predicted values are draws from the log-normal distribution whose logarithm has mean and variance equal to  $\mu_{ij}$  and  $\sigma^2$ , respectively.

### 1.0.7 Output Values

The output of each Zelig command contains useful information which you may view. You may examine the available information in `z.out` by using `slotNames(z.out)`, see the fixed effect coefficients by using `summary(z.out)$coefs`, and a default summary of information through `summary(z.out)`. Other elements available through the `$` operator are listed below.

- From the `zelig()` output stored in `summary(z.out)`, you may extract:
  - `fixef`: numeric vector containing the conditional estimates of the fixed effects.
  - `ranef`: numeric vector containing the conditional modes of the random effects.
  - `frame`: the model frame for the model.
- From the `sim()` output stored in `s.out`, you may extract quantities of interest stored in a data frame:
  - `qi$pr`: the simulated predicted values drawn from the distributions defined by the expected values.
  - `qi$ev`: the simulated expected values for the specified values of `x`.
  - `qi$fd`: the simulated first differences in the expected values for the values specified in `x` and `x1`.
  - `qi$ate.pr`: the simulated average predicted treatment effect for the treated from conditional prediction models.
  - `qi$ate.ev`: the simulated average expected treatment effect for the treated from conditional prediction models.

## How to Cite the Multi-level Least Squares Model

Matthew Owen, Olivia Lau, Kosuke Imai, and Gary King. *ls.mixed: Mixed Effects Least Squares Regression*, 2011

## How to Cite the Zelig Software Package

To cite Zelig as a whole, please reference these two sources:

Kosuke Imai, Gary King, and Olivia Lau. 2007. “Zelig: Everyone’s Statistical Software,” <http://GKing.harvard.edu/zelig>.

Imai, Kosuke, Gary King, and Olivia Lau. (2008). “Toward A Common Framework for Statistical Analysis and Development.” *Journal of Computational and Graphical Statistics*, Vol. 17, No. 4 (December), pp. 892-913.

## See also

Mixed effects linear regression is part of **lme4** package by Douglas M. Bates [1].

## References

- [1] Douglas Bates. *lme4: Fit linear and generalized linear mixed-effects models*, 2007.
- [2] Matthew Owen, Olivia Lau, Kosuke Imai, and Gary King. *ls.mixed: Mixed Effects Least Squares Regression*, 2011.