

# 1 bprobit: Bivariate Logistic Regression for Two Dichotomous Dependent Variables

Use the bivariate probit regression model if you have two binary dependent variables ( $Y_1, Y_2$ ), and wish to model them jointly as a function of some explanatory variables. Each pair of dependent variables ( $Y_{i1}, Y_{i2}$ ) has four potential outcomes, ( $Y_{i1} = 1, Y_{i2} = 1$ ), ( $Y_{i1} = 1, Y_{i2} = 0$ ), ( $Y_{i1} = 0, Y_{i2} = 1$ ), and ( $Y_{i1} = 0, Y_{i2} = 0$ ). The joint probability for each of these four outcomes is modeled with three systematic components: the marginal  $\Pr(Y_{i1} = 1)$  and  $\Pr(Y_{i2} = 1)$ , and the correlation parameter  $\rho$  for the two marginal distributions. Each of these systematic components may be modeled as functions of (possibly different) sets of explanatory variables.

## 1.0.1 Syntax

```
> z.out <- zelig(list(mu1 = Y1 ~ X1 + X2,
                    mu2 = Y2 ~ X1 + X3,
                    rho = ~ 1),
                model = "bprobit", data = mydata)
> x.out <- setx(z.out)
> s.out <- sim(z.out, x = x.out)
```

## 1.0.2 Input Values

In every bivariate probit specification, there are three equations which correspond to each dependent variable ( $Y_1, Y_2$ ), and the correlation parameter  $\rho$ . Since the correlation parameter does not correspond to one of the dependent variables, the model estimates  $\rho$  as a constant by default. Hence, only two formulas (for  $\mu_1$  and  $\mu_2$ ) are required. If the explanatory variables for  $\mu_1$  and  $\mu_2$  are the same and effects are estimated separately for each parameter, you may use the following short hand:

```
> fml <- list(cbind(Y1, Y2) ~ X1 + X2)
```

which has the same meaning as:

```
> fml <- list(mu1 = Y1 ~ X1 + X2,
+            mu2 = Y2 ~ X1 + X2,
+            rho = ~ 1)
```

You may use the function `tag()` to constrain variables across equations. The `tag()` function takes a variable and a label for the effect parameter. Below, the constrained effect of `x3` in both equations is called the `age` parameter:

```
> fml <- list(mu1 = y1 ~ x1 + tag(x3, "age"),
+            mu2 = y2 ~ x2 + tag(x3, "age"))
```

You may also constrain different variables across different equations to have the same effect.

### 1.0.3 Examples

#### 1. Basic Example

Load the data and estimate the model:

```
> data(sanction)

> z.out1 <- zelig(cbind(import, export) ~ coop + cost + target,
+               model = "bprobit", data = sanction)
```

By default, `zelig()` estimates two effect parameters for each explanatory variable in addition to the correlation coefficient; this formulation is parametrically independent (estimating unconstrained effects for each explanatory variable), but stochastically dependent because the models share a correlation parameter.

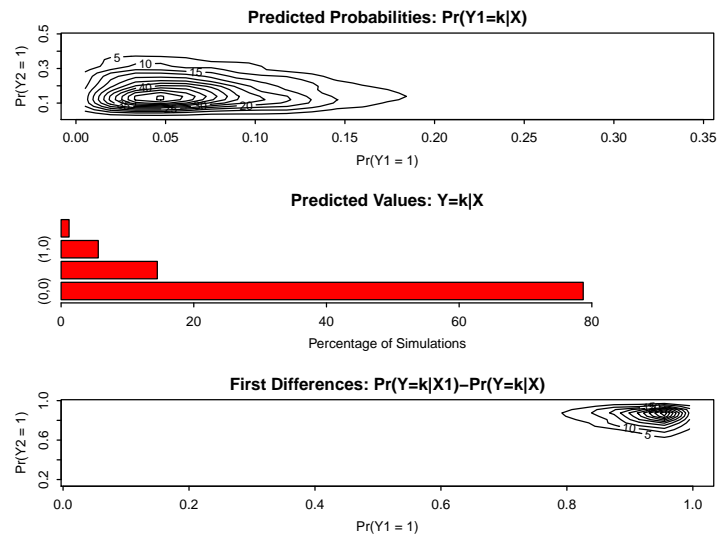
Generate baseline values for the explanatory variables (with cost set to 1, net gain to sender) and alternative values (with cost set to 4, major loss to sender):

```
> x.low <- setx(z.out1, cost = 1)
> x.high <- setx(z.out1, cost = 4)
```

Simulate fitted values and first differences:

```
> s.out1 <- sim(z.out1, x = x.low, x1 = x.high)
> summary(s.out1)

> plot(s.out1)
```



## 2. Joint Estimation of a Model with Different Sets of Explanatory Variables

Using the sample data `sanction`, estimate the statistical model, with `import` a function of `coop` in the first equation and `export` a function of `cost` and `target` in the second equation:

```
> fml2 <- list(mu1 = import ~ coop,
+             mu2 = export ~ cost + target)

> z.out2 <- zelig(fml2, model = "bprobit", data = sanction)
> summary(z.out2)
```

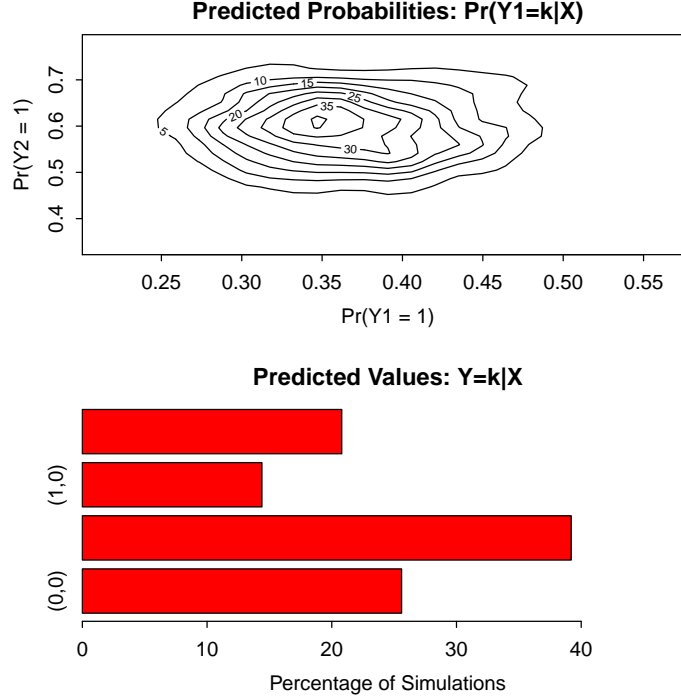
Set the explanatory variables to their means:

```
> x.out2 <- setx(z.out2)
```

Simulate draws from the posterior distribution:

```
> s.out2 <- sim(z.out2, x = x.out2)
> summary(s.out2)

> plot(s.out2)
```



#### 1.0.4 Model

For each observation, define two binary dependent variables,  $Y_1$  and  $Y_2$ , each of which take the value of either 0 or 1 (in the following, we suppress the observation index  $i$ ). We model the joint outcome  $(Y_1, Y_2)$  using two marginal probabilities for each dependent variable, and the correlation parameter, which describes how the two dependent variables are related.

- The *stochastic component* is described by two latent (unobserved) continuous variables which follow the bivariate Normal distribution:

$$\begin{pmatrix} Y_1^* \\ Y_2^* \end{pmatrix} \sim N_2 \left\{ \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right\},$$

where  $\mu_j$  is a mean for  $Y_j^*$  and  $\rho$  is a scalar correlation parameter. The following observation mechanism links the observed dependent variables,  $Y_j$ , with these latent variables

$$Y_j = \begin{cases} 1 & \text{if } Y_j^* \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

- The *systemic components* for each observation are

$$\begin{aligned}\mu_j &= x_j \beta_j \quad \text{for } j = 1, 2, \\ \rho &= \frac{\exp(x_3 \beta_3) - 1}{\exp(x_3 \beta_3) + 1}.\end{aligned}$$

### 1.0.5 Quantities of Interest

For  $n$  simulations, expected values form an  $n \times 4$  matrix.

- The expected values (**qi\$ev**) for the binomial probit model are the predicted joint probabilities. Simulations of  $\beta_1$ ,  $\beta_2$ , and  $\beta_3$  (drawn from their sampling distributions) are substituted into the systematic components, to find simulations of the predicted joint probabilities  $\pi_{rs} = \Pr(Y_1 = r, Y_2 = s)$ :

$$\begin{aligned}\pi_{11} &= \Pr(Y_1^* \geq 0, Y_2^* \geq 0) = \int_0^\infty \int_0^\infty \phi_2(\mu_1, \mu_2, \rho) dY_2^* dY_1^* \\ \pi_{10} &= \Pr(Y_1^* \geq 0, Y_2^* < 0) = \int_0^\infty \int_{-\infty}^0 \phi_2(\mu_1, \mu_2, \rho) dY_2^* dY_1^* \\ \pi_{01} &= \Pr(Y_1^* < 0, Y_2^* \geq 0) = \int_{-\infty}^0 \int_0^\infty \phi_2(\mu_1, \mu_2, \rho) dY_2^* dY_1^* \\ \pi_{00} &= \Pr(Y_1^* < 0, Y_2^* < 0) = \int_{-\infty}^0 \int_{-\infty}^0 \phi_2(\mu_1, \mu_2, \rho) dY_2^* dY_1^*\end{aligned}$$

where  $r$  and  $s$  may take a value of either 0 or 1,  $\phi_2$  is the bivariate Normal density.

- The predicted values (**qi\$pr**) are draws from the multinomial distribution given the expected joint probabilities.
- The first difference (**qi\$fd**) in each of the predicted joint probabilities are given by

$$\text{FD}_{rs} = \Pr(Y_1 = r, Y_2 = s \mid x_1) - \Pr(Y_1 = r, Y_2 = s \mid x).$$

- The risk ratio (**qi\$rr**) for each of the predicted joint probabilities are given by

$$\text{RR}_{rs} = \frac{\Pr(Y_1 = r, Y_2 = s \mid x_1)}{\Pr(Y_1 = r, Y_2 = s \mid x)}.$$

- In conditional prediction models, the average expected treatment effect (**att.ev**) for the treatment group is

$$\frac{1}{\sum_{i=1}^n t_i} \sum_{i:t_i=1}^n \{Y_{ij}(t_i = 1) - E[Y_{ij}(t_i = 0)]\} \quad \text{for } j = 1, 2,$$

where  $t_i$  is a binary explanatory variable defining the treatment ( $t_i = 1$ ) and control ( $t_i = 0$ ) groups. Variation in the simulations are due to uncertainty in simulating  $E[Y_{ij}(t_i = 0)]$ , the counterfactual expected value of  $Y_{ij}$  for observations in the treatment group, under the assumption that everything stays the same except that the treatment indicator is switched to  $t_i = 0$ .

- In conditional prediction models, the average predicted treatment effect (`att.pr`) for the treatment group is

$$\frac{1}{\sum_{i=1}^n t_i} \sum_{i:t_i=1}^n \left\{ Y_{ij}(t_i = 1) - Y_{ij}(\widehat{t_i = 0}) \right\} \text{ for } j = 1, 2,$$

where  $t_i$  is a binary explanatory variable defining the treatment ( $t_i = 1$ ) and control ( $t_i = 0$ ) groups. Variation in the simulations are due to uncertainty in simulating  $Y_{ij}(\widehat{t_i = 0})$ , the counterfactual predicted value of  $Y_{ij}$  for observations in the treatment group, under the assumption that everything stays the same except that the treatment indicator is switched to  $t_i = 0$ .

### 1.0.6 Output Values

The output of each Zelig command contains useful information which you may view. For example, if you run `z.out <- zelig(y ~ x, model = "bprobit", data)`, then you may examine the available information in `z.out` by using `names(z.out)`, see the `coefficients` by using `z.out$coefficients`, and obtain a default summary of information through `summary(z.out)`. Other elements available through the `$` operator are listed below.

- From the `zelig()` output object `z.out`, you may extract:
  - `coefficients`: the named vector of coefficients.
  - `fitted.values`: an  $n \times 4$  matrix of the in-sample fitted values.
  - `predictors`: an  $n \times 3$  matrix of the linear predictors  $x_j\beta_j$ .
  - `residuals`: an  $n \times 3$  matrix of the residuals.
  - `df.residual`: the residual degrees of freedom.
  - `df.total`: the total degrees of freedom.
  - `rss`: the residual sum of squares.
  - `y`: an  $n \times 2$  matrix of the dependent variables.
  - `zelig.data`: the input data frame if `save.data = TRUE`.
- From `summary(z.out)`, you may extract:
  - `coef3`: a table of the coefficients with their associated standard errors and  $t$ -statistics.

- `cov.unscaled`: the variance-covariance matrix.
- `pearson.resid`: an  $n \times 3$  matrix of the Pearson residuals.
- From the `sim()` output object `s.out`, you may extract quantities of interest arranged as arrays indexed by simulation  $\times$  quantity  $\times$  `x`-observation (for more than one `x`-observation; otherwise the quantities are matrices). Available quantities are:
  - `qi$ev`: the simulated expected values (joint predicted probabilities) for the specified values of `x`.
  - `qi$pr`: the simulated predicted outcomes drawn from a distribution defined by the joint predicted probabilities.
  - `qi$fd`: the simulated first difference in the predicted probabilities for the values specified in `x` and `x1`.
  - `qi$rr`: the simulated risk ratio in the predicted probabilities for given `x` and `x1`.
  - `qi$att.ev`: the simulated average expected treatment effect for the treated from conditional prediction models.
  - `qi$att.pr`: the simulated average predicted treatment effect for the treated from conditional prediction models.

## How to Cite the Bivariate Probit Model

Olivia Lau, Kosuke Imai, and Gary King. *Bivariate Probit Regression for Two Dichotomous Dependent Variables*, 2011

## How to Cite the Zelig Software Package

To cite Zelig as a whole, please reference these two sources:

Kosuke Imai, Gary King, and Olivia Lau. 2007. “Zelig: Everyone’s Statistical Software,” <http://GKing.harvard.edu/zelig>.

Imai, Kosuke, Gary King, and Olivia Lau. (2008). “Toward A Common Framework for Statistical Analysis and Development.” *Journal of Computational and Graphical Statistics*, Vol. 17, No. 4 (December), pp. 892-913.

## See also

The bivariate probit function is part of the VGAM package by Thomas Yee [3]. In addition, advanced users may wish to refer to `help(vglm)` in the VGAM library. Additional documentation is available at <http://www.stat.auckland.ac.nz/~yee>. Sample data are from [2]

## References

- [1] Olivia Lau, Kosuke Imai, and Gary King. *Bivariate Probit Regression for Two Dichotomous Dependent Variables*, 2011.
- [2] Lisa Martin. *Coercive Cooperation: Explaining Multilateral Economic Sanctions*. Princeton University Press, 1992. Please inquire with Lisa Martin before publishing results from these data, as this dataset includes errors that have since been corrected.
- [3] T. W. Yee and T. J. Hastie. Reduced-rank vector generalized linear models. *Statistical Modelling*, 3:15–41, 2003.